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STEADY-STATE HEAT TRANSFER WITH AXIAL CONDUCTION IN LAMINAR FLOW IN A CIRCULAR TUBE WITH A SPECIFIED TEMPERATURE OR HEAT FLUX WALL

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Abstract—The solution for the temperature profile is obtained on the basis of the superposition principle or by the use of Green's function when the wall temperature (or heat flux) profile is a function of axial distance in a section between two infinitely long uniform temperature (or adiabatic) wall inlet and outlet sections. It is confirmed that the solution satisfies numerically the matching conditions which are imposed on the temperature and its axial derivative at each end of the main heat transfer section. In addition, it is suggested that the expansion theory employed here can be generalized easily for a generalized Sturm–Liouville system.

NOMENCLATURE

$\mathbf{A}$,	matrix defined by equation (34);
A_m,	coefficient defined by equation (13);
a,	lower limit of y;
$a_{i,j,k}$,	coefficient in equation (A9);
a_m,	series expansion coefficient in equation (5) or (17);
a_m^*,	$= a_m \exp(-\lambda_m L)$;
B_m,	coefficient defined by equation (16) or (21);
b,	upper limit of y;
$b_{i,j,k}$,	coefficient in equation (A9);
$b_{i,m}(=b_i)$,	series expansion coefficient in equation (A1);
b_m,	series expansion coefficient in equation (5) or (17);
b_m^*,	$= b_m \exp(-\lambda_m L)$;
C_m,	coefficient defined by equation (29);
c_m,	series expansion coefficient in equation (10) or (35);
d_i,	coefficient defined by equation (A2);
$F_{\pm i}(y)$,	function of y defined by equation (28);
$f(x)$,	arbitrary function of x;
$f_i(y)$,	solution of recurrence differential equation (8);
$G(x, y)$,	Green's function;
$H_{\pm i}(y)$,	function of y defined by equation (28);
h_a, h_b,	constants in equation (31);
$h(x)$,	arbitrary function of x;
$h_i(y)$,	solution of recurrence differential equation (19);
$I_{i,j}$,	coefficient defined by equation (A4);
$J_0(\alpha_i y)$, $J_1(\alpha_i y)$,	Bessel functions;
L,	$= l/r_0\,Pe$, dimensionless length of main heat transfer section;
l,	length of main heat transfer section;
Pe,	$= 2r_0 u_m/\alpha_H$, Peclet number;
$p(y)$,	function of y;
$R_m(\lambda)$,	function of λ defined by equation (A6);
r,	radial distance;
r_0,	tube radius;
T,	dimensionless temperature;
T_M,	dimensionless cup-mixing temperature;
$u_i(y)$,	function of y;
u_m,	mean axial velocity;
$\mathbf{v}$,	arbitrary vector;
$v_i(y)$,	arbitrary function of y;
x,	$= z/r_0 Pe$, dimensionless axial distance;
Y_m,	eigenfunction;
y,	$= r/r_0$, dimensionless radial distance;
$\mathbf{z}_m$,	vector defined by equation (34);
z,	axial distance.

Greek symbols

α_H,	thermal diffusivity;
α_i,	root defined by equation (A2);
$\delta_{i,j}$,	Kronecker's delta;
$\delta(x)$,	Dirac's delta function;
$\lambda_m(=\lambda)$,	eigenvalue.

1. INTRODUCTION

THE PROBLEM of steady-state heat transfer with axial conduction in laminar flow in a circular tube has been solved analytically [1–4] or numerically [5–7] when the wall temperature profile is a step function [1, 2, 5, 6], when the wall heat flux profile is a step function [3, 5, 6], or when the wall temperature profile is uniform in an infinitely long outlet section with an infinitely long adiabatic wall inlet section [4, 6, 7]. These wall conditions are reasonable physically. In most industrial heat transfer problems, however, the wall temperature or heat flux profile is a function of axial distance, and the length of the main heat transfer section is finite. Because of this point of view, the present study aims at obtaining an exact solution to the problem of steady-state heat transfer with axial conduction in laminar flow in a circular tube with a specified temperature (or heat flux) wall in a section between two infinitely long uniform temperature (or adiabatic) wall inlet and outlet sections.

An exact solution for the temperature profile is obtained on the basis of the superposition principle when the wall temperature or heat flux profile in the main heat transfer section can be expanded in a power series of axial distance. On the other hand, a similar problem for slug flow in a coaxial annulus has been solved by use of Green's function [8], and this approach is available for solving problems of this sort whether the velocity profile is uniform or not. Hence, another solution is obtained by use of Green's function which is derived from the solution with a uniform temperature or heat flux wall condition in the main heat transfer section. It is, of course, confirmed that both solutions are identical when the wall temperature or heat flux profile in the main heat transfer section can be expanded in a power series of axial distance. It is, in addition, confirmed that the solution satisfies numerically the matching conditions which are imposed on the temperature and its axial derivative at each end of the main heat transfer section.

2. BASIC EQUATIONS

The governing heat transfer equation is, in dimensionless form, given as follows, e.g. [3]:

$$(1-y^2)\frac{\partial T}{\partial x} = \frac{1}{Pe^2}\frac{\partial^2 T}{\partial x^2} + \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right), \quad \left(\begin{array}{c} -\infty < x < +\infty \\ 0 < y < 1 \end{array}\right) \tag{1}$$

with

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0. \tag{2}$$

When the wall temperature profile is a function of axial distance in a section between two infinitely long uniform temperature wall inlet and outlet sections, the boundary conditions are given as

$$(T)_{y=1} = \begin{cases} 0, & (x<0,\ x>L), \\ f(x), & (0<x<L), \end{cases}$$

$$(T)_{x=\pm\infty} = 0, \qquad (0 \leqq y \leqq 1), \qquad \text{(Case 1)}. \tag{3}$$

When the wall heat flux profile is a function of axial distance in a section between two infinitely long adiabatic wall inlet and outlet sections, the boundary conditions are given as

$$\left(\frac{\partial T}{\partial y}\right)_{y=1} = \begin{cases} 0, & (0<x<L) \\ h(x), & (0<x<L), \end{cases}$$

$$(T)_{x=-\infty} = 0, \qquad (0 \leqq y \leqq 1),$$

$$(T)_{x=+\infty} = 4\int_0^L h(x)\mathrm{d}x, \ (0 \leqq y \leqq 1), \text{(Case 2)}. \tag{4}$$

Hence, equation (1) with equation (2) is to be solved with equation (3) or (4).

3. THEORETICAL ANALYSIS

3.1. *A solution based on the superposition principle*

The solutions for $f(x) = x^n$ $(n=0, 1, \ldots)$ are now sought in the form

$$(T)_{x<0} = \sum_{m=-1}^{-\infty} a_m \exp(-\lambda_m x) Y_m(y),$$

$$(T)_{0<x<L} = \sum_{i=0}^{n} \frac{n!}{(n-i)!} x^{n-i} f_i(y) + \sum_{m=\pm1}^{\pm\infty} b_m \exp(-\lambda_m x) Y_m(y),$$

$$(T)_{x>L} = \sum_{m=1}^{\infty} a_m \exp(-\lambda_m x) Y_m(y). \tag{5}$$

Here, λ_m and $Y_m(y)$, respectively, are the eigenvalues and eigenfunctions of the following boundary value problem:

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\left(y\frac{\mathrm{d}Y_m}{\mathrm{d}y}\right) + \lambda_m\left(\frac{\lambda_m}{Pe^2} + 1 - y^2\right)Y_m = 0,$$

$$\left(\frac{\mathrm{d}Y_m}{\mathrm{d}y}\right)_{y=0} = 0, \quad (Y_m)_{y=1} = 0, \quad (m = \pm 1, \pm 2, \ldots), \tag{6}$$

where

$$\lambda_{-m} < 0, \quad \lambda_m > 0, \quad (m = 1, 2, \ldots). \tag{7}$$

In addition, $f_i(y)$ are the solutions of the following recurrence differential equations:

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\left(y\frac{\mathrm{d}f_i}{\mathrm{d}y}\right) = (1-y^2)f_{i-1} - \frac{1}{Pe^2}f_{i-2},$$

$$\left(\frac{\mathrm{d}f_i}{\mathrm{d}y}\right)_{y=0} = 0, \quad (f_i)_{y=1} = 0, \quad (i = 1, 2, \ldots), \tag{8}$$

where

$$f_{-1}(y) = 0, \quad f_0(y) = 1. \tag{9}$$

It is evident that the solutions assumed by equation (5) satisfy equations (1), (2), and (3) for $f(x) = x^n$ $(n = 0, 1, \ldots)$. Hence, the complete solutions are determined on the basis of the matching conditions which are imposed on the temperature and its axial derivative at $x = 0$ and $x = L$. In this case, the expansion technique commonly used for the Sturm–Liouville system cannot be utilized because the present eigenfunctions lack the classical orthogonality properties. However, the following technique presented by Smith *et al.* [9] is available.

The differential system given by equation (6) has an infinite number of not only positive but also negative eigenvalues. Hence, not one but two arbitrary functions of y, $v_1(y)$ and $v_2(y)$, which are quite irrelevant to each other can be expanded as

$$v_1(y) = \sum_{m=\pm1}^{\pm\infty} c_m Y_m(y),$$

$$v_2(y) = \sum_{m=\pm1}^{\pm\infty} \lambda_m c_m Y_m(y), \quad (0<y<1). \tag{10}$$

In addition, the present eigenfunctions have the following properties which can be derived easily from equation (6):

$$\int_0^1\left(\frac{\lambda_m+\lambda_s}{Pe^2}+1-y^2\right)yY_mY_s\mathrm{d}y=0,$$

$$(m\neq s;\quad m,s=\pm 1,\pm 2,\ldots). \quad (11)$$

This allows for a term-by-term calculation of the series expansion coefficients in equation (10) as follows [9]:

$$c_m=\int_0^1\left[\left(\frac{\lambda_m}{Pe^2}+1-y^2\right)v_1(y)+\frac{1}{Pe^2}v_2(y)\right]$$

$$\times\, yY_m\mathrm{d}y/A_m,\quad (m=\pm 1,\pm 2,\ldots), \quad (12)$$

where

$$A_m=\int_0^1\left(\frac{2\lambda_m}{Pe^2}+1-y^2\right)yY_m^2\mathrm{d}y,\quad (m=\pm 1,\pm 2,\ldots). \quad (13)$$

In the present case, the following further relations are derived from equations (6), (8), and (9) with the aid of integration of parts:

$$\int_0^1\left[\left(\frac{\lambda_m}{Pe^2}+1-y^2\right)f_i-\frac{1}{Pe^2}f_{i-1}\right]yY_m\mathrm{d}y$$

$$=\frac{(\mathrm{d}Y_m/\mathrm{d}y)_{y=1}}{(-\lambda_m)^{i+1}},\quad (i=0,1,\ldots). \quad (14)$$

Hence, the series expansion coefficients in equation (5) are determined finally as

$$a_{-m}=\sum_{i=0}^{n}\frac{n!}{(n-i)!}\frac{B_{-m}L^{n-i}}{(-\lambda_{-m})^{i+1}}[\delta_{n,i}-\exp(\lambda_{-m}L)],$$

$$b^*_{-m}=\sum_{i=0}^{n}\frac{n!}{(n-i)!}\frac{B_{-m}L^{n-i}}{(-\lambda_{-m})^{i+1}},$$

$$b_m=n!\frac{B_m}{(-\lambda_m)^{n+1}},$$

$$a^*_m=\sum_{i=0}^{n}\frac{n!}{(n-i)!}\frac{B_mL^{n-i}}{(-\lambda_m)^{i+1}}[1-\delta_{n,i}\exp(-\lambda_mL)],$$

$$(m=1,2,\ldots). \quad (15)$$

where $\delta_{n,i}$ is Kronecker's delta, and

$$a^*_m=a_m\exp(-\lambda_mL),\quad b^*_{-m}=b_{-m}\exp(-\lambda_{-m}L),$$

$$B_{\pm m}=(\mathrm{d}Y_{\pm m}/\mathrm{d}y)_{y=1}/A_{\pm m},\quad (m=1,2,\ldots). \quad (16)$$

In the same way as described above, the solutions for $h(x)=x^n$ $(n=0,1,\ldots)$ are obtained as shown in Table 1, equations (17)–(21).

As mentioned above, the solutions for $f(x)$ or $h(x)=x^n$ $(n=0,1,\ldots)$ are obtained on the basis of the superposition principle. Hence, a solution for arbitrary $f(x)$ or $h(x)$ is obtained by superposing these solutions again when $f(x)$ or $h(x)$ can be expanded in a power series of x.

3.2. *Another solution based on Green's function*

Green's function, $G(x,y)$, defined as the solution for $f(x)$ or $h(x)=\delta(x)$ where $\delta(x)$ is the Dirac delta function, and is derived from the following formula [8]:

$$G(x,y)=\lim_{L\to 0}T(x,y)\ \text{for}\ f(x)\ \text{or}\ h(x)=\frac{1}{L}. \quad (22)$$

Hence, Green's function for Case 1 or Case 2 is obtained as

$$G(x<0,y)=\sum_{m=1}^{\infty}B_{-m}\exp(-\lambda_{-m}x)Y_{-m}(y),$$

$$G(x>0,y)=-\sum_{m=1}^{\infty}B_m\exp(-\lambda_mx)Y_m(y),\ \text{(Case 1)},$$

$$G(x<0,y)=-\sum_{m=1}^{\infty}B_{-m}\exp(-\lambda_{-m}x)Y_{-m}(y),$$

$$G(x>0,y)=4+\sum_{m=1}^{\infty}B_m\exp(-\lambda_mx)Y_m(y),$$

$$\text{(Case 2)}. \quad (23)$$

Further, it is evident that the following formula provides a solution for arbitrary $f(x)$ or $h(x)$ [8]:

$$T(x,y)=\int_0^L[f(\xi)\ \text{or}\ h(\xi)]G(x-\xi,y)\mathrm{d}\xi. \quad (24)$$

Hence, another solution for arbitrary $f(x)$ or $h(x)$ is obtained as shown in Table 2, equations (25) and (26).

3.3. *Comparison of both solutions*

The matching conditions which are imposed on the temperature and its axial derivative at $x=0$ and $x=L$ provide the following formulae (note that the formula for $f_0(y)$ is not valid at $y=1$):

$$F_{-i}(y)=f_{i-1}(y)-F_{+i}(y),\quad \text{(Case 1)},$$

$$H_{-i}(y)=-4h_i(y)-H_{+i}(y),\quad \text{(Case 2)},$$

$$(0\leqq y\leqq 1;\quad i=0,1,\ldots), \quad (27)$$

where

$$F_{\pm i}(y)\ \text{or}\ H_{\pm i}(y)=\sum_{m=1}^{\infty}\frac{B_{\pm m}}{(-\lambda_{\pm m})^i}Y_{\pm m}(y),$$

$$(i=0,1,\ldots). \quad (28)$$

On this basis, it is easily confirmed that both solutions mentioned above are identical when $f(x)$ or $h(x)=x^n$ $(n=0,1,\ldots)$. Hence, it is evident that both solutions for arbitrary $f(x)$ or $h(x)$ also are identical when $f(x)$ or $h(x)$ can be expanded in a power series of x.

4. CALCULATED RESULTS

The differential system given by equation (6) or (18) can be solved by means of Galerkin's method (e.g. [10]) (see Appendix I), and the first 30 positive and negative eigenvalues and their corresponding eigenfunctions were calculated for $Pe=1$–100 and ∞

Table 1. The solutions for $h(x) = x^n$ $(n = 0, 1, \ldots)$

The complete solutions:

$$(T)_{x<0} = \sum_{m=1}^{\infty} a_{-m} \exp(-\lambda_{-m} x)\, Y_{-m}(y), \quad (T)_{x>L} = \frac{4}{n+1} L^{n+1} + \sum_{m=1}^{\infty} a_m^* \exp[-\lambda_m (x-L)]\, Y_m(y),$$

$$(T)_{0<x<L} = 4 \sum_{i=0}^{n+1} \frac{n!}{(n-i+1)!} x^{n-i+1} h_i(y) - \sum_{m=1}^{\infty} \{ b_{-m}^* \exp[-\lambda_{-m}(x-L)]\, Y_{-m}(y) + b_m \exp(-\lambda_m x)\, Y_m(y) \}. \tag{17}$$

The differential system for $Y_m(y)$ ($\lambda_m > 0$ for $m > 0$, and $\lambda_m < 0$ for $m < 0$):

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\left(y \frac{\mathrm{d}Y_m}{\mathrm{d}y}\right) + \lambda_m \left(\frac{\lambda_m}{Pe^2} + 1 - y^2\right) Y_m = 0, \quad \left(\frac{\mathrm{d}Y_m}{\mathrm{d}y}\right)_{y=0} = 0, \quad \left(\frac{\mathrm{d}Y_m}{\mathrm{d}y}\right)_{y=1} = 0, \quad (m = \pm 1, \pm 2, \ldots). \tag{18}$$

The differential system for $h_i(y)$ $[h_{-1}(y) = 0$, and $h_0(y) = 1]$;

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\left(y \frac{\mathrm{d}h_i}{\mathrm{d}y}\right) = (1-y^2) h_{i-1} - \frac{1}{Pe^2} h_{i-2}, \quad \left(\frac{\mathrm{d}h_i}{\mathrm{d}y}\right)_{y=0} = 0, \quad \left(\frac{\mathrm{d}h_{i+1}}{\mathrm{d}y}\right)_{y=1} = 0, \quad (i = 1, 2, \ldots). \tag{19}$$

The series expansion coefficients in equation (17);

$$a_{-m} = -\sum_{i=0}^{n} \frac{n!}{(n-i)!} \frac{B_{-m} L^{n-i}}{(-\lambda_{-m})^{i+1}} [\delta_{n,i} - \exp(\lambda_{-m} L)], \quad b_{-m}^* = -\sum_{i=0}^{n} \frac{n!}{(n-i)!} \frac{B_{-m} L^{n-i}}{(-\lambda_{-m})^{i+1}},$$

$$a_m^* = -\sum_{i=0}^{n} \frac{n!}{(n-i)!} \frac{B_m L^{n-i}}{(-\lambda_m)^{i+1}} [1 - \delta_{n,i} \exp(-\lambda_m L)], \quad b_m = -n! \frac{B_m}{(-\lambda_m)^{n+1}}, \quad (m = 1, 2, \ldots), \tag{20}$$

where

$$B_m = (Y_m)_{y=1} \Big/ \int_0^1 \left(\frac{2\lambda_m}{Pe^2} + 1 - y^2\right) y Y_m^2 \mathrm{d}y, \quad (m = \pm 1, \pm 2, \ldots). \tag{21}$$

(note that no negative eigenvalue exists for $Pe = \infty$). Some sample values of them are shown in Table 3 where $C_{\pm m}$ represent the following coefficients which are required to calculate the cup-mixing quantities:

$$C_{\pm m} = 4 \int_0^1 (1-y^2)\, y Y_{\pm m} \mathrm{d}y, \quad (m = 1, 2, \ldots) \tag{29}$$

The differential system given by equation (8) or (19) can easily be solved in order (see Appendix II), and the first 20 solutions were determined. The first three of them are shown in Table 4, equation (30).

To check the validity of equation (27), $F_{\pm m}(y)$ or $H_{\pm m}(y)$ were calculated by employing the first 30 positive and negative eigenvalues and their corresponding eigenfunctions. In this case, it is clear from Table 3 that the direct sums for $F_{\pm 0}(0)$ [or $H_{\pm 0}(0)$] diverge (or converge slowly). In such a case, these sums were calculated with the aid of Euler's transformation. As Table 5 shows, the calculated results support the validity of equation (27), namely, the fact that the present solutions satisfy the required matching conditions.

Some sample axial temperature profiles for $f(x) = 1$ and $Pe \cdot L = 1$ are shown in Fig. 1 ($Pe = 1$) and Fig. 2

Table 2. A solution for arbitrary $f(x)$ or $h(x)$ based on Green's function

A solution for Case 1:

$$(T)_{x<0} = \sum_{m=1}^{\infty} B_{-m} Y_{-m}(y) \int_0^L f(\xi) \exp[-\lambda_{-m}(x-\xi)] \mathrm{d}\xi, \quad (T)_{x>L} = -\sum_{m=1}^{\infty} B_m Y_m(y) \int_0^L f(\xi) \exp[-\lambda_m (x-\xi)] \mathrm{d}\xi,$$

$$(T)_{0<x<L} = \sum_{m=1}^{\infty} \left\{ B_{-m} Y_{-m}(y) \int_x^L f(\xi) \exp[-\lambda_{-m}(x-\xi)] \mathrm{d}\xi - B_m Y_m(y) \int_0^x f(\xi) \exp[-\lambda_m (x-\xi)] \mathrm{d}\xi \right\}. \tag{25}$$

A solution for Case 2:

$$(T)_{x<0} = -\sum_{m=1}^{\infty} B_{-m} Y_{-m}(y) \int_0^L h(\xi) \exp[-\lambda_{-m}(x-\xi)] \mathrm{d}\xi, \quad (T)_{x>L} = 4 \int_0^L h(\xi) \mathrm{d}\xi + \sum_{m=1}^{\infty} B_m Y_m(y) \int_0^L h(\xi) \exp[-\lambda_m (x-\xi)] \mathrm{d}\xi,$$

$$(T)_{0<x<L} = 4 \int_0^x h(\xi) \mathrm{d}\xi - \sum_{m=1}^{\infty} \left\{ B_{-m} Y_{-m}(y) \int_x^L h(\xi) \exp[-\lambda_{-m}(x-\xi)] \mathrm{d}\xi - B_m Y_m(y) \int_0^x h(\xi) \exp[-\lambda_m (x-\xi)] \mathrm{d}\xi \right\}. \tag{26}$$

Table 3. Some sample eigenvalues and their corresponding eigenfunctions

m	λ_{-m}	$B_{-m}C_{-m}$	$B_{-m}Y_{-m}(0)$	λ_m	B_mC_m	$B_mY_m(0)$
(Pe = 1, Case 1)						
1	-2.825296819	1.198396396	1.946490838	2.044367805	-1.090394870	-1.865019072
2	-5.876321504	0.047962400	-3.011985658	5.187344037	-0.131583256	2.858906794
3	-8.999200814	0.017204568	3.748840450	8.323450740	-0.032769752	-3.614852901
4	-12.132922190	0.007519921	-4.359440664	11.461365119	-0.012256897	4.241474748
5	-15.270218160	0.003916943	4.893581545	14.600502245	-0.005795557	-4.787509681
6	-18.409114200	0.002290366	-5.374589837	17.740366285	-0.003174201	5.277582209
7	-21.548860083	0.001452416	5.815813919	20.880683005	-0.001920501	-5.725953836
8	-24.689110037	0.000977893	-6.225776794	24.021297527	-0.001248214	6.141720029
9	-27.829683042	0.000689404	6.610324012	27.162117437	-0.000856159	-6.531092412
10	-30.970475310	0.000504059	-6.973674810	30.303084627	-0.000612403	6.898533225
(Pe = 1, Case 2)						
1	-0.497420129	-4.02025736	-3.95802775	3.51896151	-0.127884432	-0.670288838
2	-4.18747650	0.15421007	0.61364041	6.69229402	-0.024810121	0.485425699
3	-7.35923114	0.02086588	-0.46289098	9.84684830	-0.008180993	-0.399825689
4	-10.51363651	0.00698184	0.38693342	12.99542546	-0.003607344	0.347892213
5	-13.66216276	0.00315783	-0.33929440	16.1413733	-0.001891948	-0.312087755
6	-16.8080863	0.00169077	0.30583401	19.2859383	-0.001111696	0.285478154
7	-19.9526377	0.00100930	-0.28066730	22.4296877	-0.000707524	-0.264696388
8	-23.0963788	0.00065022	0.26084759	25.5729164	-0.000477699	0.247883206
9	-26.2396019	0.00044323	-0.24471354	28.7157924	-0.000337525	-0.233916677
10	-29.3824740	0.00031558	0.23124746	31.8584185	-0.000247233	0.222074219
(Pe = 10, Case 1)						
1	-74.767174	15.023882	8.0693495	6.744048933	-5.191258484	-9.306439446
2	-103.90065	-7.298478	-24.192085	30.767918207	-2.100393724	20.922996501
3	-128.93675	1.311657	39.133948	59.503463202	-0.873505642	-29.341380841
4	-157.58366	-0.091190	-46.743543	89.476677963	-0.378694205	36.376232247
5	-187.52746	0.000414	52.326352	119.97261248	-0.177008213	-42.475034043
6	-218.01950	0.001032	-57.12745	150.75037263	-0.090425105	47.88330553
7	-248.80081	0.001924	61.45626	181.69864606	-0.050299462	-52.77092908
8	-279.75391	0.002021	-65.45043	212.75772822	-0.030130571	57.2552073
9	-310.81740	0.001864	69.18707	243.89269650	-0.019199782	-61.4180604
10	-341.9560	0.001640	-72.71542	275.08181599	-0.012873054	65.3181990
(Pe = 10, Case 2)						
1	-37.16544	-4.384436	-1.246671	18.78794265	-0.332491227	-5.913241447
2	-90.5368	1.368829	1.663630	45.43800310	-0.217767961	5.074856877
3	-115.1829	-0.296891	-2.817626	74.52727490	-0.106259913	-4.298799120
4	-142.4600	0.052200	3.043088	104.64084529	-0.052264679	3.735785660
5	-172.0226	0.014117	-2.887614	135.23611865	-0.027743463	-3.329769878
6	-202.3683	0.009437	2.708599	166.08835661	-0.015983339	3.02584923
7	-233.0831	0.006223	-2.546517	197.09291050	-0.009890858	-2.78942128
8	-264.0034	0.004304	2.405368	228.1951571	-0.006492240	2.59948762
9	-295.0502	0.003092	-2.282953	259.3638590	-0.004470487	-2.44290679
10	-326.1805	0.002292	2.176228	290.5798502	-0.003200899	2.3111043

(Pe = 10); those for $h(x) = Pe$ and $Pe \cdot L = 1$, in Fig. 3 (Pe = 1 and 10); and some sample cup-mixing temperature profiles, in Fig. 4 [$f(x) = 1$] and Fig. 5 [$h(x) = Pe$]. In either case, it can be observed that the fluid in the entrance section ($x < 0$) is preheated more easily for smaller Peclet number flow, and the fluid temperature in the exit section ($x > L$) approaches its terminal temperature more slowly for larger Peclet number flow.

5. DISCUSSION

The present results for $f(x)$ or $h(x) = 1$ and $L = \infty$ or for $h(x) = \delta(x)$ were compared with the tabulated results of Jones [1, 2, 11] and with the graphical results of Hsu [3], Hennecke [5], and Verhoff and Fisher [6]. In this case, it was confirmed that the present respective results are in good agreement with the cup-mixing temperature profiles for $f(x) = 1$ in [1, 2, 5, 6], for $h(x) = 1$ in [5], and for $h(x) = \delta(x)$ in [11], with the radial temperature profiles at $x = 0$ for $f(x) = 1$ in [5] and for $h(x) = \delta(x)$ in [11], and with the Nusselt number profiles for $f(x) = 1$ and $Pe \cdot x > 0.1$ in [1, 2, 5, 6] and for $h(x) = 1$ in [3, 5]. It was observed, however, that the present Nusselt numbers for $f(x) = 1$ and

Table 4. The first three solutions of equation (8) or (19).

$$f_1(y) = -\frac{3}{16} + \frac{1}{4}y^2 - \frac{1}{16}y^4$$

$$f_2(y) = \frac{251}{9216} - \frac{3}{64}y^2 + \frac{7}{256}y^4 - \frac{5}{576}y^6 + \frac{1}{1024}y^8 + \frac{1}{Pe^2}\left(\frac{1}{4} - \frac{1}{4}y^2\right)$$

$$f_3(y) = -\frac{4627}{1228800} + \frac{251}{36864}y^2 - \frac{683}{147456}y^4 + \frac{19}{9216}y^6 - \frac{83}{147456}y^8 + \frac{89}{921600}y^{10} - \frac{1}{147456}y^{12} - \frac{1}{Pe^2}\left(\frac{41}{576} + \frac{7}{64}y^2 - \frac{3}{64}y^4 + \frac{5}{576}y^6\right)$$

$$h_1(y) = -\frac{7}{96} + \frac{1}{4}y^2 - \frac{1}{16}y^4 + \frac{2}{Pe^2}$$

$$h_2(y) = \frac{9}{2560} - \frac{7}{384}y^2 + \frac{31}{1536}y^4 - \frac{5}{576}y^6 + \frac{1}{1024}y^8 + \frac{1}{Pe^2}\left(0 + \frac{1}{4}y^2 - \frac{1}{8}y^4\right) + \frac{4}{Pe^4}$$

$$h_3(y) = -\frac{45659}{309657600} + \frac{9}{10240}y^2 - \frac{167}{122880}y^4 + \frac{59}{55296}y^6 - \frac{133}{294912}y^8 + \frac{89}{921600}y^{10} - \frac{1}{147456}y^{12} - \frac{1}{Pe^2}\left(\frac{9}{1280} - \frac{7}{384}y^2 + 0y^4 + \frac{5}{576}y^6 - \frac{1}{512}y^8\right) + \frac{1}{Pe^4}\left(\frac{1}{24} + \frac{1}{2}y^2 - \frac{1}{4}y^4\right) + \frac{8}{Pe^6}. \quad (30)$$

Table 5. Confirmation of the validity of equation (27)

	i	Pe	Cup-mixing value	$y=0$	$y=0.25$	$y=0.5$	$y=0.75$
$\frac{f_{i-1}(y) - F_{+i}(y)}{F_{-i}(y)} - 1$	0	1	-9.2×10^{-6}	5.3×10^{-8}	2.7×10^{-3}	-7.6×10^{-9}	9.3×10^{-6}
		10	-1.3×10^{-4}	2.9×10^{-6}	-1.4×10^{-1}	-2.3×10^{-7}	6.5×10^{-6}
	1	1	9.1×10^{-6}	3.9×10^{-11}	-1.6×10^{-4}	-3.7×10^{-8}	3.0×10^{-4}
		10	2.9×10^{-5}	-1.1×10^{-8}	-1.2×10^{-3}	-2.2×10^{-7}	9.6×10^{-5}
	2	1	-6.3×10^{-9}	1.2×10^{-10}	-3.8×10^{-5}	1.1×10^{-9}	1.6×10^{-6}
		10	-4.7×10^{-7}	4.1×10^{-8}	-1.5×10^{-3}	-1.9×10^{-7}	1.2×10^{-5}
$\frac{-4h_i(y) - H_{+i}(y)}{H_{-i}(y)} - 1$	0	1	5.4×10^{-7}	9.3×10^{-9}	1.8×10^{-4}	-1.6×10^{-7}	-1.4×10^{-7}
		10	6.5×10^{-5}	3.7×10^{-6}	1.1×10^{-3}	5.9×10^{-6}	1.5×10^{-4}
	1	1	-2.6×10^{-7}	-1.5×10^{-10}	-2.2×10^{-6}	-2.7×10^{-9}	-9.1×10^{-7}
		10	-2.0×10^{-5}	-7.2×10^{-8}	1.8×10^{-5}	-5.3×10^{-8}	-5.0×10^{-5}
	2	1	1.9×10^{-11}	-1.1×10^{-10}	-6.0×10^{-9}	-2.5×10^{-11}	6.7×10^{-11}
		10	7.9×10^{-8}	6.8×10^{-8}	3.6×10^{-6}	5.5×10^{-8}	-6.4×10^{-6}

$Pe \cdot x < 0.1$ are lower than those obtained by Hennecke [5] and higher than those obtained by Jones [2], and that the present radial temperature profiles at $x=0$ for $h(x) = 1$ are higher than those obtained by Hennecke [5].

The form of Hsu's solution [3] is identical with that of the present one for $h(x) = 1$ and $L = \infty$, and he determined the series expansion coefficients with the aid of a Gram–Schmidt orthonormalization procedure. This technique, however, is unnecessarily complicated in comparison with the present one which allows for a term by term calculation of the series expansion coefficients. On the other hand, Jones analyzed the problem for $h(x) = 1$ and $L = \infty$ [1, 2] or $h(x) = \delta(x)$ [11] by employing a two-sided Laplace transform. Although he obtained only the asymptotic solution at low [2, 11] or high [1] Peclet number, it is confirmed that his solution is equivalent to the present one.

The most significant part of the present analysis is in the expansion theory given by equations (10)–(13). Agrawal [12] seems to be the first to employ the expansion technique given by equation (12) though he did not describe explicitly. After that, Deavours [13]

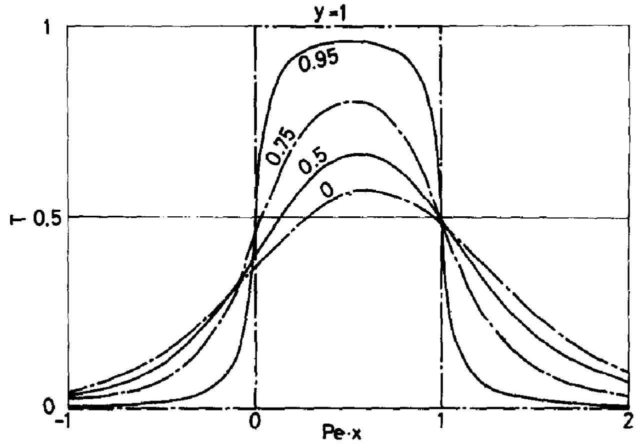

FIG. 1. Some sample axial temperature profiles for $f(x) = 1$ and $Pe \cdot L = 1$ ($Pe = 1$).

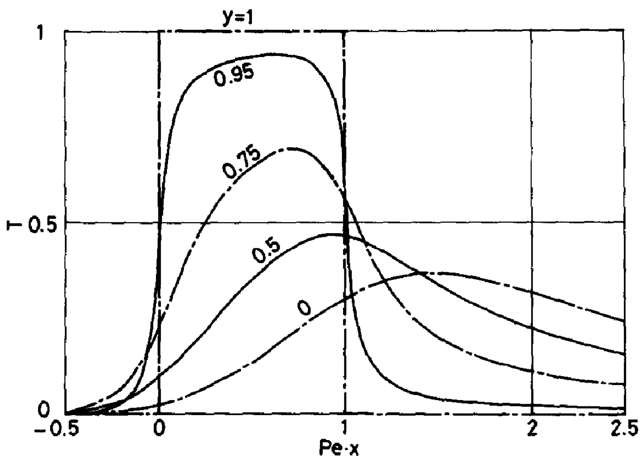

FIG. 2. Some sample axial temperature profiles for $f(x) = 1$ and $Pe \cdot L = 1$ ($Pe = 10$).

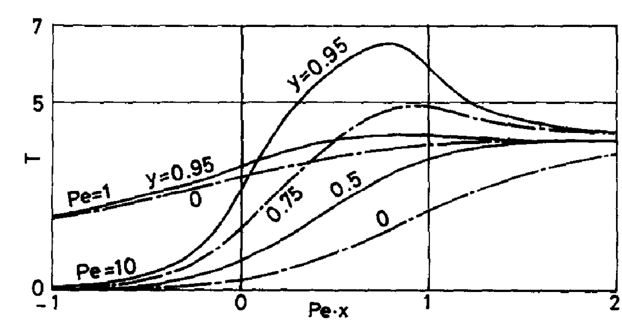

FIG. 3. Some sample axial temperature profiles for $h(x) = Pe$ and $Pe \cdot L = 1$ ($Pe = 1$ and 10).

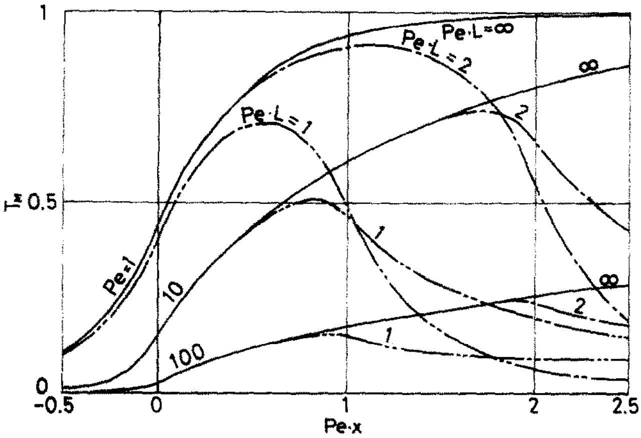

FIG. 4. Some sample cup-mixing temperature profiles [$f(x) = 1$].

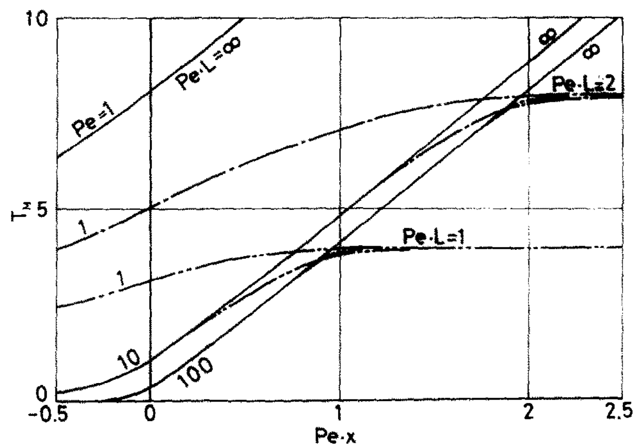

FIG. 5. Some sample cup-mixing temperature profiles [$h(x) = Pe$].

developed an expansion theory which is essentially equivalent to that developed by Smith *et al.* [9]. However, the latter is more intended for practical use. Hence, the latter was employed here. Further, this theory can be generalized easily as follows.

The differential system given by equation (6) or (18) is generalized as

$$\frac{d}{dy}\left\{p(y)\frac{dY_m}{dy}\right\} + \sum_{i=0}^{N} \lambda_m^i u_i(y)\, Y_m = 0, \quad (a<y<b),$$

$$\left(Y_m + h_a\frac{dY_m}{dy}\right)_{y=a} = 0, \quad \left(Y_m + h_b\frac{dY_m}{dy}\right)_{y=b} = 0,$$

$$(m = 1, 2, \ldots). \quad (31)$$

In this case, it can be confirmed easily that the eigenfunctions have the following properties:

$$\int_a^b \left[\sum_{i=1}^{N}\left(\sum_{j=1}^{i} \lambda_m^{i-j}\lambda_s^{j-1}\right)u_i(y)\right] Y_m Y_s \,dy = 0,$$

$$(m \neq s;\ m, s = 1, 2, \ldots), \quad (32)$$

or

$$\int_a^b \mathbf{z}_m^T \mathbf{A} \mathbf{z}_s \,dy = 0, \quad (m \neq s;\ m, s = 1, 2, \ldots), \quad (33)$$

where

$$\mathbf{A} = (a_{i,j}),\ a_{i,j} = \begin{cases} u_{i+j-1}(y), & (i+j-1 \leqq N), \\ 0, & (i+j-1 > N), \end{cases}$$

$$\mathbf{z}_m = (Y_m, \lambda_m Y_m, \ldots, \lambda_m^{N-1} Y_m)^T, \quad (m = 1, 2, \ldots). \quad (34)$$

Equation (33) indicates that the eigenvectors, $\mathbf{z}_m$, are mutually orthogonal over the interval, (a, b), with respect to the matrix, **A**. Hence, an arbitrary vector, **v**, can be expanded as

$$\mathbf{v} = (v_1(y), v_2(y), \ldots, \mathbf{v}_N(y))^T = \sum_{m=1}^{\infty} c_m \mathbf{z}_m, \quad (35)$$

where

$$c_m = \int_a^b \mathbf{z}_m^T \mathbf{A} \mathbf{v} \,dy \Big/ \int_a^b \mathbf{z}_m^T \mathbf{A} \mathbf{z}_m \,dy, \quad (m = 1, 2, \ldots). (36)$$

In the previous studies, the differential system given by equation (6) or (18) has been solved by several methods [1–4, 7, 11, 14]. These methods, however, are not suitable for obtaining the sufficiently higher eigenvalues with a good accuracy. On the other hand, the present method which is equivalent to Galerkin's method is comparatively suitable for the above purpose.

The analysis based on Green's function becomes a failure when the wall temperature (or heat flux) profile is a function of axial distance in a section between two infinitely long adiabatic (or uniform temperature) wall inlet and outlet sections. On the contrary, the analysis based on the superposition principle holds for this problem though the series expansion coefficients can not be calculated term by term. However, the solution for the present problem with arbitrary $f(x)$ or $h(x)$ is obtained more easily by use of the former than the latter. Hence, it is difficult to choose the better of the two.

6. CONCLUDING REMARKS

(1) The analysis based on Green's function is very simple, but the application is narrow.

(2) The analysis based on the superposition principle is somewhat complicated, but the application is wide.

(3) Galerkin's method is comparatively suitable for obtaining the sufficiently higher eigenvalues with a good accuracy.

(4) The expansion technique developed by Smith *et al.* more intended for practical use, and can be generalized for a generalized Sturm–Liouville system.

(5) The length of the main heat transfer section, as well as the axial conduction term, is a significant factor for problems of this sort.

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APPENDIX I

The eigenfunctions for the system given by equation (6) or (18) can be expanded in the following infinite series of Bessel functions which satisfy the boundary conditions in equation (6) or (18) [15]:

$$Y_m = \sum_{i=1}^{\infty} b_{i,m} J_0(\alpha_i y)/d_i, \quad (m = \pm 1, \pm 2, \ldots), \tag{A1}$$

where

$$\begin{aligned} &\alpha_i = \text{the } i\text{th root of } J_0(\alpha)=0, \ (0<\alpha_1<\alpha_2<\ldots), \\ &d_i = J_1(\alpha_i), \ (i=1,2,\ldots), \qquad \text{(Case 1)}; \\ &\alpha_i = \text{the } i\text{th root of } J_1(\alpha)=0, \ (0=\alpha_1<\alpha_2<\ldots), \\ &d_i = J_0(\alpha_i), \ (i=1,2,\ldots), \qquad \text{(Case 2)}. \end{aligned} \tag{A2}$$

In this case, the differential system given by equation (6) or (18) is reduced to the following infinite set of linear homogeneous simultaneous equations for $b_{j,m}$[15]:

$$\sum_{j=1}^{\infty} \left[\left(\frac{\lambda_m}{Pe^2} - \frac{\alpha_i^2}{\lambda_m}\right)\delta_{i,j} + I_{i,j}\right] b_{j,m} = 0, \quad (i=1,2,\ldots), \tag{A3}$$

where

$$I_{i,j} = \frac{2}{d_i d_j}\int_0^1 (1-y^2)\, y J_0(\alpha_i y) J_0(\alpha_j y)\,\mathrm{d}y, \quad (i,j = 1,2,\ldots). \tag{A4}$$

Hence the following condition for non-vanishing $b_{j,m}$ gives an infinite number of positive and negative eigenvalues, and equation (A3) gives their corresponding $b_{j,m}$ if one of $b_{j,m}$ is replaced arbitrarily by 1 [15]:

$$\det\left|\left(\frac{\lambda_m}{Pe^2} - \frac{\alpha_i^2}{\lambda_m}\right)\delta_{i,j} + I_{i,j}\right| = 0. \tag{A5}$$

In practice, however, the infinite series in equation (A1) must be truncated at the Mth term. Hence, this method for obtaining the eigenfunctions becomes quite equivalent to Galerkin's method.

The eigenvalues can be calculated directly from equation (A5) by means of some iterative methods, e.g. the regulafalsi method [10] or the Newton–Raphson method, but these methods take a comparatively long computational time. Hence, it is better to proceed as follows. If equation (A3) is rearranged with $b_{m,m} = 1$, $\lambda_m = \lambda$, and $b_{j,m} = b_j$, it becomes the following equations for $i = m$ and $i \neq m$:

$$R_m(\lambda) \equiv \frac{\lambda}{Pe^2} - \frac{\alpha_m^2}{\lambda} + I_{m,m} + \sum_{\substack{j=1\\(j\neq m)}}^{M} I_{m,j} b_j = 0, \tag{A6}$$

$$\sum_{\substack{j=1\\(j\neq m)}}^{M}\left[\left(\frac{\lambda}{Pe^2}-\frac{\alpha_i^2}{\lambda}\right)\delta_{i,j}+I_{i,j}\right]b_j+I_{i,m}=0,$$

$$(i\neq m;\ i=1,2,\ldots,M). \quad \text{(A7)}$$

Equation (A7) is a finite set of linear simultaneous equations for b_j and can be solved for b_j if λ is known. Hence, $R_m(\lambda)$ defined by equation (A6) is considered to be a function of λ alone, and the eigenvalues are given as the roots of $R_m(\lambda)=0$. In the present study, the first 30 positive and negative eigenvalues for $M=60$ were determined by means of the Newton–Raphson method. In this case, the approximate eigenvalues which are determined by means of the WKB method were used as the initial estimates of λ, and the mth positive and negative eigenvalues were usually determined from $R_m(\lambda)=0$. However, the mth positive or negative eigenvalue was rarely determined from $R_{m-1}(\lambda)=0$ when the following conditions are not satisfied:

$$\frac{\lambda^{(2)}-\lambda^{(1)}}{\lambda^{(1)}-\lambda^{(0)}}<1 \text{ and } \frac{R_m(\lambda^{(1)})}{R_m(\lambda^{(0)})}<1, \quad \text{(A8)}$$

where $\lambda^{(i)}$ is the ith estimate of λ_m. This is due to the fact that the root of $R_m(\lambda)=0$ which is determined by means of the above method converges not the mth eigenvalue but the sth one in such a case.

APPENDIX II

The solutions for the system given by equation (8) or (19) are now sought of the form

$$f_i(y)=\sum_{j=1}^{[i/2]+1}\left(\frac{1}{Pe^2}\right)^{j-1}\sum_{k=1}^{2i-3j+4}a_{i,j,k}\,y^{2k-2},$$

$$h_i(y)=\sum_{j=1}^{i+1}\left(\frac{1}{Pe^2}\right)^{j-1}\sum_{k=1}^{2i-2j+3}b_{i,j,k}\,y^{2k-2},$$

$$(i=0,1,\ldots), \quad \text{(A9)}$$

where $[i/2]$ is Gauss' notation and

$$a_{0,1,1}=b_{0,1,1}=1. \quad \text{(A10)}$$

Substitution of this into equation (8) or (19) gives the following recurrence equations for $a_{i,j,k}$ or $b_{i,j,k}$:

$$a_{i,j,k}=(a_{i-1,j,k-1}-a_{i-1,j,k-2}-a_{i-2,j-1,k-1})/(2k-2)^2,$$

$$(k=2,3,\ldots,2i-3j+4),$$

$$a_{i,j,1}=-\sum_{k=2}^{2i-3j+4}a_{i,j,k},$$

$$(i=1,2,\ldots;\ j=1,2,\ldots,[i/2]+1),$$

$$b_{i,j,k}=(b_{i-1,j,k-1}-b_{i-1,j,k-2}-b_{i-2,j-1,k-1})/(2k-2)^2,$$

$$(k=2,3,\ldots,2i-2j+3),$$

$$b_{i,j,1}=2b_{i-1,j-1,1}-\sum_{k=2}^{2i-2j+3}2\left(\frac{b_{i,j,k}}{k+1}-b_{i-1,j-1,k}\right)/k,$$

$$(i=1,2,\ldots;\ j=1,2,\ldots,i+1), \quad \text{(A11)}$$

where $a_{i,j,k}$ or $b_{i,j,k}$ which appear in equation (A11) but do not appear in equation (A9) are zero. Hence, the solutions given by equation (A9) are determined on the basis of equations (A10) and (A11).

TRANSFERT THERMIQUE STATIONNAIRE AVEC CONDUCTION AXIALE DANS UN ECOULEMENT LAMINAIRE DANS UN TUBE CIRCULAIRE AVEC UNE PAROI A TEMPERATURE OU A FLUX THERMIQUE DONNE

Résumé—La solution du profil de température est obtenue par le principe de superposition ou par utilisation de la fonction de Green quand le profil de température de paroi (ou de flux thermique) est une fonction de la distance axiale de la section deux zones infiniment longues avec une paroi à température uniforme (ou adiabatique). La solution satisfait numériquement les conditions imposées sur la température et sur la dérivée axiale aux extrémités de la zone principale de transfert thermique. De plus, il est suggéré que la méthode de développement utilisée ici peut-être étendue facilement au système généralisé de Sturm–Liouville.

STATIONÄRER WÄRMEDURCHGANG MIT AXIALER WÄRMELEITUNG BEI LAMINARER STRÖMUNG IN EINEM KREISROHR MIT VORGEGEBENER TEMPERATUR ODER WÄRMESTROMDICHTE DER WAND

Zusammenfassung — Die Berechnung des Temperaturprofils wird nach dem Superpositionsprinzip oder mit Hilfe der Green'schen Funktion durchgeführt, wobei das Wandtemperaturprofil (oder die Wärmestromdichte) eine Funktion der axialen Koordinate zwischen zwei unendlich langen Ein- und Auslaufabschnitten mit gleichförmiger Temperatur der Wand (oder mit adiabater Wand) ist. Es wird festgestellt, daß die Lösung zahlenmäßig sehr gut die Anschlußbedingungen erfüllt, die für die Temperatur und ihre Ableitung in axialer Richtung an beiden Enden der Hauptwärmeübergangszone vorgegebenen sind. Zusätzlich wird gezeigt, daß die hier angewandte Reihenentwicklung leicht für die Anwendung auf ein generelles Sturm–Liouville-System verallgemeinert werden kann.

СТАЦИОНАРНЫЙ ПЕРЕНОС ТЕПЛА ПРИ АКСИАЛЬНОЙ ТЕПЛОПРОВОДНОСТИ И ЛАМИНАРНОМ ТЕЧЕНИИ В КРУГЛОЙ ТРУБЕ С ЗАДАННОЙ ТЕМПЕРАТУРОЙ ИЛИ ПЛОТНОСТЬЮ ТЕПЛОВОГО ПОТОКА НА СТЕНКЕ

Аннотация — С помощью принципа суперпозиции или функций Грина получено решение для температурного профиля в случае, когда распределение температуры стенки (или теплового потока) является функцией расстояния по оси в области между двумя бесконечно удаленными входным и выходным сечениями, находящимися при однородной (или адиабатической) температуре стенки. Численно установлено, что решение удовлетворяет условиям сопряжения, налагаемым на температуру и ее аксиальную производную с двух сторон основного участка теплообмена. Кроме того, высказано предположение, что используемый метод можно легко обобщить на систему Штурма–Лиувилля.